Study material of B.Sc.(Semester - I)
US01CMTH02
(Differential Equations -1)

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US01CMTH02
UNIT-4

1. Differential equations

1.1. Definition. An equation involving derivatives of a dependent variable with respect to one or more independent variables and is called a *differential equation.*

The following are examples of differential equations.

(1) $\frac{dy}{dx} - 2x\cos x = 0.$

(2) $x^2\left(\frac{d^2y}{dx^2}\right)^3 + y\left(\frac{dy}{dx}\right)^5 + xy = 0.$

(3) $y - x\frac{dy}{dx} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{5/2}.$

(4) $\frac{d^2x}{dy^2} + \left[1 + \left(\frac{dx}{dy}\right)^3\right]^{\frac{1}{2}} = 0.$

(5) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z.$

(6) $\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial z}{\partial y}.$

If a differential equation contains one dependent variable (or function) and one independent variable, that is, if all the derivatives appearing have reference to the same single independent variable, then the differential equation is called an *ordinary differential equation.* If there are two or more independent variables, so that the derivatives are partial, then the equation is called a *partial differential equation.* Thus, the differential equations (1) to (4) above are ordinary differential equations and (5) and (6) are partial differential equations. The *order of a differential equation* is the order of the highest differential coefficient which appears in it. The *degree of a differential equation* is the highest degree of the highest order derivative appearing in it, when **all** the derivatives are free from radicals and fractional powers. For example, in the above the differential equation (1) is of first order and of first degree; (2) is of second order and of third degree; (Observe that there is a fifth power term in the equation however, it is the power of first order derivative). Equation (3) is of first order and of tenth degree. Equation (4) is of second order and of second degree; (5) is of first order and of first degree; and (6) is of second order and of first degree. Note that in equation (3), to make the highest order differential $\frac{dy}{dx}$ independent of any rational power, we have to take the square of both the sides, which in turn, will give rise the highest power 10 of the differential coefficient $\frac{dy}{dx}$.

Exact Differential Equations.

Let $\mathbf{M}$ and $\mathbf{N}$ be functions of x and y. A differential equation of the form

$$\mathbf{M}dx + \mathbf{N}dy = 0, \tag{1.1.1}$$

is said to be *exact* if the expression on the left hand side of (1.1.1) can be obtained directly by differentiating some function of x and y.

1.2. Theorem. *The necessary and sufficient condition for the differential equation*

$$\mathbf{M}dx + \mathbf{N}dy = 0 \tag{1.2.1}$$

to be exact is that

$$\frac{\partial \mathbf{M}}{\partial y} = \frac{\partial \mathbf{N}}{\partial x}. \tag{1.2.2}$$

PROOF. **Necessity:** Suppose that (1.2.1) is exact. So, $\mathbf{M}dx + \mathbf{N}dy$ can be obtained directly by differentiating some function $f = f(x, y)$. Thus,

$$d[f(x,y)] = \mathbf{M}dx + \mathbf{N}dy \Rightarrow \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \mathbf{M}dx + \mathbf{N}dy.$$

So,

$$\frac{\partial f}{\partial x} = \mathbf{M} \quad \text{and} \quad \frac{\partial f}{\partial y} = \mathbf{N};$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial \mathbf{M}}{\partial y} \text{ and } \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial \mathbf{N}}{\partial x}. \tag{1.2.3}$$

Since, we assume the function to be many times continuously differentiable, (see Proposition **??**), $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$. As a result, (1.2.3) gives $\frac{\partial \mathbf{M}}{\partial y} = \frac{\partial \mathbf{N}}{\partial x}$.

Sufficiency. Let $\mathbf{P} = \int \mathbf{M}dx$. Then $\frac{\partial \mathbf{P}}{\partial x} = \mathbf{M}$. Hence $\frac{\partial^2 \mathbf{P}}{\partial y \partial x} = \frac{\partial \mathbf{M}}{\partial y}$. This together with (1.2.2) gives, $\frac{\partial \mathbf{N}}{\partial x} = \frac{\partial \mathbf{M}}{\partial y} = \frac{\partial^2 \mathbf{P}}{\partial y \partial x} = \frac{\partial^2 \mathbf{P}}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \mathbf{P}}{\partial y}\right)$. This, on integrating with respect to x, gives,

$$\mathbf{N} = \frac{\partial \mathbf{P}}{\partial y} + \varphi(y),$$

where φ is a function of y only. Thus we have,

$$\begin{aligned}
\mathbf{M}dx + \mathbf{N}dy &= \frac{\partial \mathbf{P}}{\partial x}dx + \left[\frac{\partial \mathbf{P}}{\partial y} + \varphi(y)\right] dy \\
&= \frac{\partial \mathbf{P}}{\partial x}dx + \frac{\partial \mathbf{P}}{\partial y}dy + \varphi(y)dy \\
&= d\mathbf{P} + d(\mathbf{F}(y)) \qquad (\text{where } d(\mathbf{F}(y)) = \varphi(y)dy) \\
&= d[\mathbf{P} + \mathbf{F}(y)],
\end{aligned}$$

which shows that $\mathbf{M}dx + \mathbf{N}dy = 0$, is an exact differential equation. □

A working rule for solving an exact differential equation

Let us assume that the differential equation (1.1.1) is exact. The following algorithm describes a working rule to solve the equation.

(1) Integrate $\mathbf{M}$ with respect to x regarding y as a constant.

(2) Integrate with respect to y, the terms in $\mathbf{N}$ not involving x.

(3) Add the two expressions obtained in the above two steps and equate the result to an arbitrary constant. This gives the required solution.

1.3. Example. Solve $(x^2 - 2xy - y^2)dx - (x + y)^2 dy = 0$.

SOLUTION. Here $\mathbf{M} = x^2 - 2xy - y^2$ and $\mathbf{N} = -(x + y)^2 = -x^2 - 2xy - y^2$. Hence,

$$\frac{\partial \mathbf{M}}{\partial y} = -2x - 2y \quad \text{and} \quad \frac{\partial \mathbf{N}}{\partial x} = -2x - 2y.$$

Thus $\frac{\partial \mathbf{M}}{\partial y} = \frac{\partial \mathbf{N}}{\partial x}$. Hence the given equation is an exact equation. Integrating $\mathbf{M}$ with respect to x treating y as constant, we obtain

$$\int \mathbf{M} dx = \int (x^2 - 2xy - y^2) dx = \frac{x^3}{3} - x^2 y - y^2 x. \tag{1.3.1}$$

Also, there is only one term $-y^2$ in $\mathbf{N}$ that does not involve x which, on integration, gives

$$\int -y^2 dy = -\frac{y^3}{3}. \tag{1.3.2}$$

Adding the right hand side terms of (1.3.1) and (1.3.2) and equating the result with a constant gives the solution $\frac{x^3}{3} - x^2 y - y^2 x - \frac{y^3}{3} = C$, where C is constant of integration. □

1.4. Example. Solve $xdx + ydy + \frac{xdy - ydx}{x^2 + y^2} = 0$.

SOLUTION. First of all we have to simplify the given equation to express it in the form of (1.1.1).

$$\begin{aligned} & xdx + ydy + \frac{xdy - ydx}{x^2 + y^2} = 0 \\ \Rightarrow \quad & \left[x - \frac{y}{x^2 + y^2} \right] dx + \left[y + \frac{x}{x^2 + y^2} \right] dy = 0. \end{aligned}$$

Thus,

$$\begin{aligned} & \mathbf{M} = x - \frac{y}{x^2 + y^2} \qquad \text{and} \qquad \mathbf{N} = y + \frac{x}{x^2 + y^2} \\ \Rightarrow \quad & \frac{\partial \mathbf{M}}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \qquad \text{and} \qquad \frac{\partial \mathbf{N}}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2}. \end{aligned}$$

Hence the given equation is exact. Now integrating $\mathbf{M}$ with respect to x regarding y as constant, we get,

$$\begin{aligned} \int \mathbf{M} dx &= \int \left[x - \frac{y}{x^2 + y^2} \right] dx \\ &= \int x dx - y \int \frac{1}{x^2 + y^2} dx \end{aligned}$$

$$= \frac{x^2}{2} - y\frac{1}{y}\tan^{-1}\left(\frac{x}{y}\right)$$
$$= \frac{x^2}{2} - \tan^{-1}\left(\frac{x}{y}\right). \tag{1.4.1}$$

Also, $N = y + \frac{x}{x^2+y^2}$. The only term in $\mathbf{N}$ not involving x is y, on integrating it with respect to y gives $\frac{y^2}{2}$. Adding this term to the last term of (1.4.1) and equating the sum to the constant gives the solution $\frac{x^2}{2} - \tan^{-1}\frac{x}{y} + \frac{y^2}{2} = C$. That is, $x^2 - 2\tan^{-1}\frac{x}{y} + y^2 = C$, where C is arbitrary constant. □

1.5. Example. Solve $(y^2e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0$.

SOLUTION. Here $\mathbf{M} = y^2e^{xy^2} + 4x^3$ and $\mathbf{N} = 2xye^{xy^2} - 3y^2$. Therefore, we have $\frac{\partial \mathbf{M}}{\partial y} = 2ye^{xy^2} + y^2 2xye^{xy^2} = \frac{\partial \mathbf{N}}{\partial x}$, ensuring that the given equation is exact. Integrating $\mathbf{M}$ with respect to x treating y as constant, we obtain

$$\int \mathbf{M}dx = \int (y^2e^{xy^2} + 4x^3)dx = y^2\frac{1}{y^2}e^{xy^2} + x^4 = e^{xy^2} + x^4.$$

Also, $-3y^2$ is the only term of $\mathbf{N}$ free from x whose integral is $-y^3$. Hence the required solution is $e^{xy^2} + x^4 - y^3 = C$, with constant C.

Alternative method. Note that the given equation can be written as

$$\begin{aligned} 0 &= (y^2e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy \\ &= y^2e^{xy^2}dx + 4x^3dx + 2xye^{xy^2}dy - 3y^2dy \\ &= y^2e^{xy^2}dx + 2xye^{xy^2}dy + 4x^3dx - 3y^2dy \\ &= d(e^{xy^2}) + d(x^4) - d(y^3), \end{aligned}$$

which, on integration, gives

$$C = e^{xy^2} + x^4 - y^3,$$

the solution of the given equation. □

Differential Equations of Higher degree

2. Introduction

Usually $\frac{dy}{dx}$ is denoted by p in differential equations which involve $\frac{dy}{dx}$ in degree greater than one. Thus the general form of a *first order n^{th} degree differential equation* is

$$p^n + A_1p^{n-1} + A_2p^{n-2} + \cdots + A_{n-1}p + A_n = 0, \tag{2.0.1}$$

where $p = \frac{dy}{dx}$ and $A_1, A_2, \ldots, A_n$ are functions of x and y. We discuss various methods to solve such equations.

3. Equations solvable for p

Splitting up the left hand side of (2.0.1) into n linear factors, we have,

$$[p - f_1(x, y)][p - f_2(x, y)] \cdots [p - f_n(x, y)] = 0.$$

Equating each factor to zero gives a differential equation of the first order and first degree which can be easily solved by using various methods discussed in Chapter **??**. Suppose the solutions of these equations are given by

$$F_1(x, y, C_1) = 0,\ F_2(x, y, C_2) = 0, \ldots, F_n(x, y, C_n) = 0. \tag{3.0.1}$$

Then the solution of (2.0.1) can be written in the form

$$F_1(x, y, C_1)F_2(x, y, C_2) \cdots F_n(x, y, C_n) = 0. \tag{3.0.2}$$

Here the arbitrary constants $C_1, C_2, \ldots, C_n$ have been replaced by a single arbitrary constant C, as every particular solution obtained from (3.0.1) can be obtained from (3.0.2) by assigning a particular value of C.

3.1. Example. Solve the following differential equations.

$$(p + y + x)(xp + x + y)(p + 2x) = 0.$$

Solution. Here we get $p + y + x = 0$ or $xp + x + y = 0$ or $p + 2x = 0$. If $p + y + x = 0$, then $\frac{dy}{dx} + x + y = 0$. Let $v = x + y$. Then,

$$\begin{aligned} &\frac{dv}{dx} - 1 + v = 0 \\ \Rightarrow\ &\frac{dv}{dx} = 1 - v \\ \Rightarrow\ &\frac{dv}{1 - v} = dx \\ \Rightarrow\ &- \log(1 - v) = x + C_1 \\ \Rightarrow\ &1 - v = e^{-x-C_1} = Ce^{-x} \\ \Rightarrow\ &1 - x - y - Ce^{-x} = 0. \end{aligned} \tag{3.1.1}$$

If $xp + x + y = 0$, then $x\frac{dy}{dx} + y = -x$, that is, $\frac{dy}{dx} + \frac{1}{x}y = -1$, which is linear and its integrating factor is $e^{\int \frac{dx}{x}} = e^{\log x} = x$. And its solution is

$$\begin{aligned} y \times \text{I.F.} &= \int -1 \times \text{I.F.}dx + C_2 \\ \Rightarrow yx &= -\int xdx + C_2 \\ \Rightarrow yx &= -\frac{x^2}{2} + C_2 \\ \Rightarrow 2xy &+ x^2 - C = 0, \quad (C = 2C_2). \end{aligned} \tag{3.1.2}$$

If $p + 2x = 0$, then $\frac{dy}{dx} + 2x = 0$, that is, $dy + 2xdx = 0$. Hence,

$$y + x^2 - C = 0. \tag{3.1.3}$$

From (3.1.1), (3.1.2) and (3.1.3), the required solution is

$$(1 - x - y - Ce^{-x})(2xy + x^2 - C)(y + x^2 - C) = 0.$$

□

3.2. Example. Solve $p^2 - xy = y^2 - px$.

Solution. The given equation can be rewritten as

$$\begin{aligned}(p^2 - y^2) + (px - xy) = 0 &\Rightarrow (p - y)(p + y + x) = 0 \\ &\Rightarrow p - y = 0 \text{ or } p + y + x = 0.\end{aligned}$$

If $p - y = 0$, then

$$\begin{aligned}&\Rightarrow \frac{dy}{dx} - y = 0 \\ &\Rightarrow \frac{dy}{y} = dx \\ &\Rightarrow \log y = x + \log C \\ &\Rightarrow \log\left(\frac{y}{C}\right) = x \\ &\Rightarrow y = Ce^x \\ &\Rightarrow y - Ce^x = 0. \qquad (3.2.1)\end{aligned}$$

If $p + y + x = 0$, then $\frac{dy}{dx} + y = -x$, which is linear in y and its integrating factor is $e^{\int dx} = e^x$. Hence its solution is

$$\begin{aligned}ye^x &= -\int xe^x dx + C \\ \Rightarrow ye^x &= -(xe^x - \int e^x dx) + C \\ \Rightarrow ye^x &= -xe^x + e^x + C \\ \Rightarrow y &= -x + 1 + Ce^{-x} \\ \Rightarrow y &+ x - 1 - Ce^{-x} = 0. \qquad (3.2.2)\end{aligned}$$

Hence from (3.2.1) and (3.2.2), the required solution is

$$(y - Ce^x)(y + x - 1 - Ce^{-x}) = 0.$$

□

3.3. Example. Solve $p^2 + 2py\cot x = y^2$.

Solution. Given equation can be written as
$p^2 + (2y\cot x)p - y^2 = 0$.
Solving this for p, we get,

$$\begin{aligned}p &= \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x + 4y^2}}{2} \\ \Rightarrow p &= -y\cot x \pm y\,\text{cosec}\, x\end{aligned}$$

$$\Rightarrow p = y\left(\frac{-\cos x \pm 1}{\sin x}\right)$$
$$\Rightarrow p = y\left(\frac{-\cos x + 1}{\sin x}\right) \text{ or } p = y\left(\frac{-\cos x - 1}{\sin x}\right)$$
$$\Rightarrow p = \frac{2y\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} \text{ or } p = -\frac{2y\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$
$$\Rightarrow p = y\tan\frac{x}{2} \text{ or } p = -y\cot\tfrac{x}{2}.$$

If $p = y\tan\frac{x}{2}$, then

$$\frac{dy}{y} = \tan\tfrac{x}{2}\,dx$$
$$\Rightarrow \log y = 2\log\left(\sec\tfrac{x}{2}\right) + \log C_1$$
$$\Rightarrow y = C_1\sec^2\tfrac{x}{2}$$
$$\Rightarrow y\cos^2\tfrac{x}{2} = C_1$$
$$\Rightarrow y(1+\cos x) = 2C_1 = C, \text{(say)}$$
$$\Rightarrow y(1+\cos x) - C = 0.$$

If $p = -y\cot\frac{x}{2}$, then

$$\frac{dy}{y} = -\cot\tfrac{x}{2}dx$$
$$\Rightarrow \log y = -2\log\left(\sin\tfrac{x}{2}\right) + \log C_2$$
$$\Rightarrow y = \frac{C_2}{\sin^2\frac{x}{2}}$$
$$\Rightarrow y\sin^2\tfrac{x}{2} = C_2$$
$$\Rightarrow y(1-\cos x) = 2C_2 = C, \text{(say)}$$
$$\Rightarrow y(1-\cos x) - C = 0.$$

So, the required solution is $(y(1+\cos x) - C)(y(1-\cos x) - C) = 0$. □

4. Equations solvable for y

This type of equation can be put in the form

$$y = f(x, p). \tag{4.0.1}$$

Differentiating (4.0.1) with respect to x gives $p = \frac{dy}{dx} = \varphi\left(x, p, \frac{dp}{dx}\right)$, which is a differential equation involving two variables x and p and hence its solution will be of the form

$$F(x, p, C) = 0, \tag{4.0.2}$$

where C is an arbitrary constant. We can now eliminate p from (4.0.1), (4.0.2) and obtain the required solution. In case elimination of p is not possible, we may solve (4.0.1) and (4.0.2) for x, y to obtain $x = F_1(p, C)$, $y = F_2(p, C)$ as the required solution, where p is the parameter.

4.1. Example. Solve $y = 2px + p^n$.

SOLUTION. Differentiating with respect to x, we get,

$$\begin{aligned}
\frac{dy}{dx} &= 2p + 2x\frac{dp}{dx} + np^{n-1}\frac{dp}{dx} \\
\Rightarrow \; -p &= (2x + np^{n-1})\frac{dp}{dx} \\
\Rightarrow p\frac{dx}{dp} &+ 2x + np^{n-1} = 0 \\
\Rightarrow \frac{dx}{dp} &+ \frac{2}{p}x = -np^{n-2},
\end{aligned}$$

which is linear in x and its integrating factor is

$$e^{\int \frac{2}{p}dp} = e^{2\log p} = e^{\log p^2} = p^2.$$

Hence, its solution is

$$\begin{aligned}
x \times (\text{I.F.}) &= \int -np^{n-2} \times (\text{I.F.})dp + C \\
\Rightarrow xp^2 &= -n\int p^n dp + C \\
\Rightarrow xp^2 &= -\frac{np^{n+1}}{n+1} + C \\
\Rightarrow x &= -\frac{np^{n-1}}{n+1} + \frac{C}{p^2}. \qquad (4.1.1)
\end{aligned}$$

Substituting this value of x in the given differential equation we get,

$$\begin{aligned}
y &= \frac{-2pnp^{n-1}}{n+1} + \frac{2pC}{p^2} + p^n \\
\Rightarrow y &= \frac{2C}{p} - \frac{n-1}{n+1}p^n. \qquad (4.1.2)
\end{aligned}$$

The required solution is obtained by eliminating p from (4.1.1) and (4.1.2). □

4.2. Example. Solve $y = \sin p - p\cos p$.

SOLUTION. Differentiating the given equation with respect to x we get,

$$\begin{aligned}
p = \frac{dy}{dx} &= (\cos p - \cos p + p\sin p)\frac{dp}{dx} \\
\Rightarrow \; \sin p\, dp &= dx \\
\Rightarrow \; -\cos p &= x + C_1 \\
\Rightarrow \; \cos p &= C - x, \qquad \text{where } C = -C_1.
\end{aligned}$$

From the given equation we get,

$$p\cos p = \sin p - y$$

$$\Rightarrow p = \frac{\sqrt{1-\cos^2 p} - y}{\cos p}$$
$$\Rightarrow \cos^{-1}(C-x) = \frac{\sqrt{1-(C-x)^2} - y}{C-x} \qquad (\text{as } \cos p = C-x)$$
$$\Rightarrow C - x = \cos\left[\frac{\sqrt{1-(C-x)^2} - y}{C-x}\right],$$

which is the required solution. □

4.3. Example. Solve $y = yp^2 + 2px$.

Solution. The given equation can be rewritten as

$$y(1-p^2) = 2px \text{ or } y = \frac{2px}{1-p^2}. \tag{4.3.1}$$

Differentiating this with respect to x we get,

$$p = \frac{(1-p^2)(2p + 2x(dp/dx)) - 2px(-2p)(dp/dx)}{(1-p^2)^2}$$
$$\Rightarrow p(1-p^2)^2 = 2p(1-p^2) + [2x(1-p^2) + 4p^2x]\frac{dp}{dx}$$
$$\Rightarrow p(1-p^2)(1-p^2-2) = 2x(1-p^2+2p^2)\frac{dp}{dx}$$
$$\Rightarrow -p(1-p^2)(1+p^2) = 2x(1+p^2)\frac{dp}{dx}$$
$$\Rightarrow p(p^2-1) = 2x\frac{dp}{dx}$$
$$\Rightarrow \frac{2dp}{p(p^2-1)} = \frac{dx}{x}$$
$$\Rightarrow \left[\frac{1}{p-1} + \frac{1}{p+1} - \frac{2}{p}\right] dp = \frac{dx}{x}$$
$$\Rightarrow \log(p-1) + \log(p+1) - 2\log p = \log x + \log C$$
$$\Rightarrow \frac{p^2-1}{p^2} = Cx$$
$$\Rightarrow p^2 - 1 = p^2 Cx$$
$$\Rightarrow p^2(1-Cx) = 1$$
$$\Rightarrow p^2 = \frac{1}{1-Cx}$$
$$\Rightarrow p = \frac{1}{\sqrt{1-Cx}}.$$

Substituting this value of p in (4.3.1), we get,

$$y = \frac{2x/\sqrt{1-Cx}}{1-\frac{1}{1-Cx}} = \frac{2x\sqrt{1-Cx}}{-Cx}$$
$$\Rightarrow 2x\sqrt{1-Cx} + Cxy = 0,$$

which is the required solution. □

5. Clairaut's equation

The equation

$$y = px + f(p) \tag{5.0.1}$$

is known as *Clairaut's equation*. To solve it, we differentiate it with respect to x and get,

$$\begin{aligned} p &= p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx} \\ \Rightarrow & [x + f'(p)]\frac{dp}{dx} = 0 \\ \Rightarrow & \frac{dp}{dx} = 0 \text{ (hence } p = C, \text{ a constant)} \end{aligned} \tag{5.0.2}$$

or

$$x + f'(p) = 0. \tag{5.0.3}$$

Now eliminating p from (5.0.1) and (5.0.2) gives

$$y = Cx + f(C), \tag{5.0.4}$$

as a solution of (5.0.1). Hence the solution of the Clairaut's equation is obtained on replacing p by a constant C. If we eliminate p between (5.0.1) and (5.0.3), we get a solution which does not contain any arbitrary constant and is not a particular case of (5.0.4). This type of solution is known as *singular solution*. Some equations can be reduced to Clairaut's form by a suitable substitution.

5.1. Example. Solve $y^2 - 2pxy + p^2(x^2 - 1) = m^2$.

Solution. The given equation can be written as

$$(y - px)^2 = m^2 + p^2 \Rightarrow y - px = \pm\sqrt{m^2 + p^2} \Rightarrow y = px \pm \sqrt{m^2 + p^2},$$

which is in Clairaut's form. Hence by substituting $p = C$, in the given equation we get the required solution as $y^2 - 2Cxy + C^2(x^2 - 1) = m^2$. □

5.2. Example. Solve $\sin px \cos y = \cos px \sin y + p$.

Solution. The given equation can be written as

$$\begin{aligned} \sin px \cos y - \cos px \sin y = p \Rightarrow & \sin(px - y) = p \\ & \Rightarrow px - y = \sin^{-1} p \\ & \Rightarrow y = px - \sin^{-1} p, \end{aligned}$$

which is in Clairaut's form. Hence by substituting $p = C$, in the given equation we get the required solution as $\sin Cx \cos y = \cos Cx \sin y + C$. □

5.3. Example. Solve $xyp^2 - (x^2 + y^2 - 1)p + xy = 0$.

SOLUTION. Let $x^2 = u$ and $y^2 = v$, that is, $x = \sqrt{u}$ and $y = \sqrt{v}$. Hence $dx = \frac{1}{2}u^{-1/2}du$ and $dy = \frac{1}{2}v^{-1/2}dv$. This gives $p = \frac{dy}{dx} = \frac{\sqrt{u}}{\sqrt{v}}\frac{dv}{du}$. Substituting these values in the given equation we get,

$$\begin{aligned}
&\sqrt{uv}\,\frac{u}{v}\left(\frac{dv}{du}\right)^2 - (u+v-1)\sqrt{\frac{u}{v}}\,\frac{dv}{du} + \sqrt{uv} = 0\\
&\Rightarrow u\left(\frac{dv}{du}\right)^2 - (u+v-1)\frac{dv}{du} + v = 0\\
&\Rightarrow uP^2 - (u+v-1)P + v = 0, \qquad \text{where } P = \frac{dv}{du}\\
&\Rightarrow u(P^2 - P) - v(P-1) + P = 0\\
&\Rightarrow v(P-1) = uP(P-1) + P\\
&\Rightarrow v = Pu + \frac{P}{P-1}, \qquad (5.3.1)
\end{aligned}$$

which is in Clairaut's form. Hence by substituting $P = C$, in (5.3.1), we get the solution as $v = Cu + \frac{C}{C-1}$. Now substituting back the values of u and v, we get the required solution $y^2 = Cx^2 + \frac{C}{C-1}$. □

5.4. Example. Solve $y^2(y - px) = x^4p^2$.

SOLUTION. Let $x = \frac{1}{u}$, $y = \frac{1}{v}$. Then $dx = \frac{-1}{u^2}du$, $dy = \frac{-1}{v^2}dv$, $p = \frac{dy}{dx} = \frac{u^2}{v^2}\frac{dv}{du}$. Substituting these values in the given equation we get,

$$\begin{aligned}
&\frac{1}{v^2}\left[\frac{1}{v} - \frac{u^2}{v^2}\frac{dv}{du}\frac{1}{u}\right] = \frac{1}{u^4}\frac{u^4}{v^4}\left(\frac{dv}{du}\right)^2\\
&\Rightarrow \frac{1}{v^3} - \frac{u}{v^4}\frac{dv}{du} = \frac{1}{v^4}\left(\frac{dv}{du}\right)^2\\
&\Rightarrow v - u\frac{dv}{du} = \left(\frac{dv}{du}\right)^2\\
&\Rightarrow v = Pu + P^2 \qquad \text{where } P = \frac{dv}{du},
\end{aligned}$$

which is in Clairaut's form and so its solution is $v = Cu + C^2$. After substituting the values of u and v back, the solution becomes $C^2xy + Cy - x = 0$. □

5.5. Example. Solve $e^{2x}(p-1) + p^3e^{2y}e^{-x} = 0$.

SOLUTION. Let $e^x = u$ and $e^y = v$. Then $e^x dx = du$ and $e^y dy = dv$. Hence $\frac{e^y}{e^x}\frac{dy}{dx} = \frac{dv}{du}$, *i.e.*, $p = \frac{u}{v}\frac{dv}{du}$. Substituting these values in the given equation, we get,

$$\begin{aligned}
&u^2\left[\frac{u}{v}\frac{dv}{du} - 1\right] + \left[\frac{u}{v}\frac{dv}{du}\right]^3\frac{v^2}{u} = 0\\
&\Rightarrow \frac{u^3}{v}\frac{dv}{du} - u^2 + \frac{u^2}{v}\left(\frac{dv}{du}\right)^3 = 0\\
&\Rightarrow u\frac{dv}{du} - v + \left(\frac{dv}{du}\right)^3 = 0
\end{aligned}$$

$$\Rightarrow v = Pu + P^3, \quad \text{where } P = \frac{dv}{du},$$

which is in Clairaut's form. Its solution is $v = Cu + C^3$ or $e^y = Ce^x + C^3$. □

6. Trajectories of a family of curves

A curve that cuts every member of a given family of curves according to some given law is called *trajectory of the given family of curves.* Here we shall consider only the case when each trajectory cuts every member of the given family at a constant angle. If the constant angle is a right angle, then the trajectory is called an *orthogonal trajectory*

6.1. Theorem. *Obtain orthogonal trajectories of the family of curves* $f(x, y, C) = 0$ *at a constant angle.*

PROOF. Suppose that the trajectories cut every member of the family

$$f(x, y, C) = 0 \tag{6.1.1}$$

at a constant angle α. Differentiating (6.1.1) with respect to x and eliminating C between equation (6.1.1) and its derivative, we get the differential equation of the given family. Let it be

$$\varphi\left(x, y, \frac{dy}{dx}\right) = 0. \tag{6.1.2}$$

Let (X, Y) be the current coordinates of any point on the required trajectory. Then the slope of its tangent at this point is $\frac{dY}{dX}$. At a point of intersection of any member of family (6.1.2), with the trajectory, we have,

$$x = X, \qquad y = Y \tag{6.1.3}$$

and

$$\tan\alpha = \frac{dy/dx - dY/dX}{1 + (dy/dx)(dY/dX)}.$$

Hence,

$$\frac{dy}{dx} = \frac{dY/dX + \tan\alpha}{1 - \tan\alpha(dY/dX)}. \tag{6.1.4}$$

From (6.1.2), (6.1.3) and (6.1.4), x, y and $\frac{dy}{dx}$ can be eliminated and we get a relation

$$\varphi\left(X, Y, \frac{dY/dX + \tan\alpha}{1 - \tan\alpha(dY/dX)}\right) = 0, \tag{6.1.5}$$

which is the differential equation of the required family of trajectories. Solving (6.1.5), we shall obtain the cartesian equation of the family of trajectories.

□

6.2. Orthogonal Trajectories (Cartesian coordinates): Suppose that the trajectories cut every member of the family (6.1.1) at the constant angle $\frac{\pi}{2}$. Hence the tangents to both of these should be perpendicular to each other, *i.e.*, in other words,

$$\frac{dy}{dx}\frac{dY}{dX} = -1 \quad \text{or} \quad \frac{dy}{dx} = -\frac{dX}{dY}.$$

Hence the differential equation of the family of the orthogonal trajectories is $\varphi(X, Y, -\frac{dX}{dY}) = 0$. In usual notations, we see that the differential equation of the family of orthogonal trajectories of the given family of $\varphi(x, y, \frac{dy}{dx}) = 0$ is $\varphi(x, y, -\frac{dx}{dy}) = 0$, so that it is obtained on replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$.

6.3. Example. Find the orthogonal trajectories of the semi-cubical parabolas $ay^2 = x^3$, where a is the variable parameter.

Solution. Differentiating the given equation with respect to x, we get

$$2ay\frac{dy}{dx} = 3x^2 \Rightarrow 2ay^2\frac{dy}{dx} = 3x^2y \Rightarrow 2x^3\frac{dy}{dx} = 3x^2y \Rightarrow 2x\frac{dy}{dx} = 3y, \tag{6.3.1}$$

which is the differential equation of the given family. Putting $-\frac{dx}{dy}$ in place of $\frac{dy}{dx}$ in (6.3.1), we get,

$$2x(-\frac{dx}{dy}) = 3y, \Rightarrow 2xdx + 3ydy = 0 \Rightarrow x^2 + \frac{3}{2}y^2 = C,$$

which the equation of the family of the orthogonal trajectories of the given semi-cubical parabolas. □

Note: The following example deals with the asymptote. The topic is presently out of the scope of this text book. If the students have not developed the theory of asymptotes and the curve tracing, this example may safely, skipped.

6.4. Example. Find the orthogonal trajectories of family of parabolas

$$y^2 = 4a(x + a), \tag{6.4.1}$$

where a is the parameter.

Solution. Differentiating (6.4.1) with respect to x, we have

$$2y\frac{dy}{dx} = 4a.$$

Putting this value in (6.4.1), we get,

$$y^2 = 2y\frac{dy}{dx}\left[x + \frac{y}{2}\frac{dy}{dx}\right] \Rightarrow y = 2x\frac{dy}{dx} + y\left(\frac{dy}{dx}\right)^2, \tag{6.4.2}$$

which is the differential equation of the given family of parabolas. Putting $\frac{dy}{dx} = -\frac{dx}{dy}$ in (6.4.2), we get the following differential equation of the required orthogonal trajectories of the given family.

$$y = -2x\frac{dx}{dy} + y\left(-\frac{dx}{dy}\right)^2 \Rightarrow y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0,$$

which is same as the differential equation (6.4.2) of the given family of parabolas. Hence the given family (6.4.1) is self orthogonal. That is, the orthogonal trajectories of the system belong to the system itself. Hence the required equation of the orthogonal trajectories of the given family is $y^2 = 4C(x + C)$, where C is the parameter. □

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